STEPs Math Quiz 3 Review Solutions

Wednesday, August 6, 2025

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1. Differentiate each of the following functions (find $f'(x)$ or $\frac{dy}{dx}$ for each part)

   (a) $f(x) = 3x^5 - 4x^3 + 2x^2 + 7x - 2$

   Solution: $f'(x) = 15x^4 - 12x^2 + 4x + 7$

   (b) $f(x) = \frac{x^3 - 3x^2}{x}$

   Solution: $f'(x) = 2x - 3$ (first simplify the function, then differentiate)

   (c) $f(x) = \sqrt{x} + \frac{1}{x}$

   Solution: $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

   (d) $f(x) = \frac{2x-1}{x^2+1}$

   Solution: $f'(x) = \frac{-2x^2+2x+2}{(x^2+1)^2}$ (use the quotient rule)

   (e) $f(x) = \frac{\ln(x)}{x^2}$

   Solution: $f'(x) = \frac{1-2\ln(x)}{x^3}$ (use the quotient rule)

   (f) $f(x) = e^{-7x}$

   Solution: $f'(x) = -7e^{-7x}$ (use the chain rule)

   (g) $f(x) = x^2 \cdot e^x$

   Solution: $f'(x) = 2xe^x + x^2e^x = e^x(2x + x^2)$ (use the product rule)

   (h) $f(x) = (2x^2 + 3)^6$

   Solution: $f'(x) = 24x(2x^2 + 3)^5$ (use the chain rule)

   (i) $f(x) = \ln(3x^2 + 6x + 10)$

   Solution: $f'(x) = \frac{6x+6}{3x^2+6x+10}$ (use the chain rule)

   (j) $f(x) = \sqrt{5x + \frac{1}{x}}$

   Solution: $f'(x) = \frac{5 - \frac{1}{x^2}}{2\sqrt{5x + \frac{1}{x}}}$ (use the chain rule)

   (k) $f(x) = e^{3x^2+2x}$

   Solution: $f'(x) = (6x + 2)e^{3x^2+2x}$ (use the chain rule)

   (l) $f(x) = (x^2 - 3)(x + 4)^3$

Solution: $f'(x) = (x+4)^2(5x^2+8x-9)$ (use the product rule and chain rule)

(m) $f(x) = 2^{4x^2-2}$

Solution: $f'(x) = 8x \cdot 2^{4x^2-2} \cdot \ln(2)$ (use the chain rule)

(n) $f(x) = \frac{1}{\sqrt{4x^2+2}}$

Solution: $f'(x) = \frac{-4x}{(4x^2+2)^{3/2}}$ (use the chain rule and power rule)

(o) $x^2y + y^2 = 7$

Solution: $\frac{d}{dx}[x^2y+y^2] = \frac{d}{dx}[7] \implies 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2xy}{x^2+2y}$

(p) $y^2\ln(x) = x^2$

Solution: $\frac{d}{dx}[y^2\ln(x)] = \frac{d}{dx}[x^2] \implies 2y\frac{dy}{dx}\cdot\ln(x) + y^2\cdot\frac{1}{x} = 2x \implies \frac{dy}{dx} = \frac{2x-\frac{y^2}{x}}{2y\ln(x)} = \frac{2x^2-y^2}{2xy\ln(x)}$
(use chain rule and implicit differentiation)

2. A balloon's radius is growing over time according to $r(t) = 3t^2+1$.
   (a) Find the rate of change of the radius at time t.
   (b) At what time is the radius increasing at a rate of 30 cm/s?
   (c) If the volume of the balloon is $V = \frac{4}{3}\pi r^3$, express $\frac{dV}{dt}$ using the chain rule.

Solution:

(a) $r'(t) = 6t$

(b) $6t = 30 \implies t = 5$

(c) $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 4\pi r^2\cdot r'(t) = 4\pi r^2\cdot 6t = 24\pi r^2 t$

3. For a certain rectangle the length of one side is always three times the length of the other side.
   (a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?
   (b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

Solution:

(a) Let the longer side be x and the shorter side be y. Then $x = 3y$. Then, $\frac{dx}{dt} = 3\cdot\frac{dy}{dt} = 3\cdot(-2) = -6$ inches/minute.

(b) $y = 6 \implies x = 18$ and $A = x\cdot y = 3y^2 = 3\times 6^2 = 108$ square inches. Then, $\frac{dA}{dt} = \frac{dA}{dy}\cdot\frac{dy}{dt} = 6y\cdot\frac{dy}{dt} = 36\cdot(-2) = -72$ square inches/minute.

4. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters?

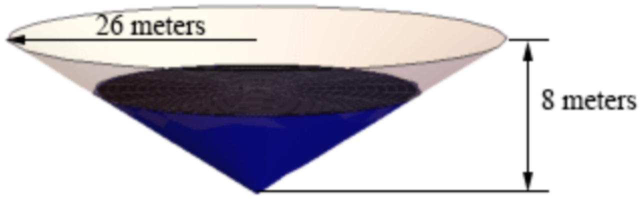

Figure 1: Water tank exercise

Solution:

(a) Let the radius of the water be r and the height of the water be h. Then, $\frac{r}{h} = \frac{26}{8} \implies r = \frac{13}{4}h$.
Also, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{13}{4}h\right)^2 h = \frac{169}{48}\pi h^3$. Then, $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = 3 \cdot \frac{169}{48}\pi h^2 \cdot \frac{dh}{dt} = 12$ (given by the problem to be 12 m^3/sec).
Solving for $\frac{dh}{dt}$, we get $\frac{dh}{dt} = \frac{12}{\frac{169}{16}\pi h^2} = \frac{192}{169\pi h^2}$.
When $r = 10$, $h = \frac{40}{13}$ and $\frac{dh}{dt} = \frac{192}{169\pi\left(\frac{40}{13}\right)^2} = \frac{192}{169\pi \cdot \frac{1600}{169}} = \frac{192}{1600\pi} = \frac{3}{25\pi} \approx 0.038$ m/sec.

5. Find two positive numbers whose product is 810 and for which the sum of one and 10 times the other is a minimum.

Solution:

(a) Let the two numbers be x and y. Then, $xy = 810$ and $x + 10y$ is minimized.
We can express y in terms of x as $y = \frac{810}{x}$. Then, $x + 10y = x + 10 \cdot \frac{810}{x} = x + \frac{8100}{x}$.
To find the minimum, we take the derivative of $x + \frac{8100}{x}$ and set it to 0.
$\frac{d}{dx}[x + \frac{8100}{x}] = 1 - \frac{8100}{x^2} = 0 \implies x^2 = 8100 \implies x = 90$.
When $x = 90$, $y = \frac{810}{90} = 9$.
The minimum sum is $90 + 10 \cdot 9 = 180$.

6. We have 48 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

Solution: Let the side length of the square base be x and the height of the box be y. Then, $x^2 + 4xy = 48$ and $V = x^2 y$ is maximized.
We can express y in terms of x as $y = \frac{48-x^2}{4x}$. Then, $V = x^2 \cdot \frac{48-x^2}{4x} = \frac{48x-x^3}{4}$.
To find the maximum, we take the derivative of $\frac{48x-x^3}{4}$ and set it to 0.
$\frac{d}{dx}[\frac{48x-x^3}{4}] = 12 - \frac{3x^2}{4} = 0 \implies 48 - 3x^2 = 0 \implies x^2 = 16 \implies x = 4$.
When $x = 4$, $y = \frac{48-16}{16} = 2$.
The maximum volume is $4^2 \cdot 2 = 32$ m^3.