STEPs Math Quiz 1 Solutions

Monday, July 14, 2025

8:00 AM - 9:00 AM version

Solutions

1. From a standard 52-card deck, we are drawing two cards *without replacement*:
   (A standard deck has 4 shapes: ♡ hearts, ♢ diamonds, ♣ clubs, ♠ spades, and 13 numbers: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K).

   (a) What is the probability the first card is a face card (J, Q, or K)? **3/13**

   (b) Given the first card drawn was a face card, what is the probability the second card is also a face card? **11/51**

2. Two island bird populations grow exponentially at different rates. Species X doubles every 3 years; Species Y doubles every 5 years. Currently, both populations have the same number of birds. After how many years will Species X population be twice that of Species Y?

   ⇒ number of years: **7.5**

Solution:

$$N \cdot 2^{t/3} = 2 \cdot N \cdot 2^{t/5}$$

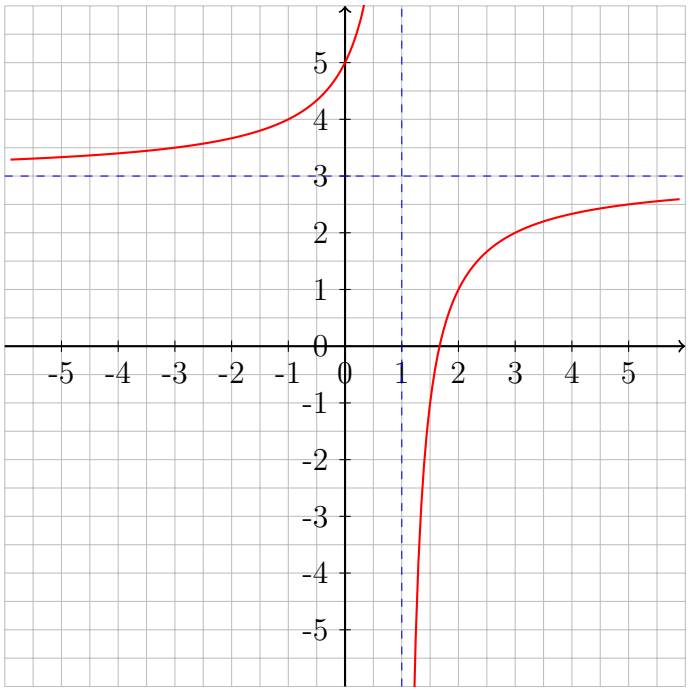
$$\Rightarrow \frac{t}{3} = \frac{t}{5} + 1$$

$$\Rightarrow \boxed{t = \frac{15}{2} = 7.5}$$

3. **Sketch** the graph of

$$r(x) = -\frac{2}{x-1} + 3$$

on the grid below, and state: the **domain** and **range**, the vertical and horizontal **asymptotes** (in the form $x = a$ or $y = b$) based on the graph.



Solution:

- Domain: $x \in \mathbb{R} \setminus \{1\} \quad \Leftrightarrow x \in (-\infty, 1) \cup (1, \infty)$
- Range: $y \in \mathbb{R} \setminus \{3\} \quad \Leftrightarrow y \in (-\infty, 3) \cup (3, \infty)$
- Vertical asymptote: $x = 1$
- Horizontal asymptote: $y = 3$

4. Let

$$f(x) = x^2 - x \quad \text{and} \quad k(x) = \sqrt{x-2}.$$

(a) Evaluate $(f \circ k)(3) =$ **0**

(b) i. Compute $(k \circ f)(x) =$ $\boxed{k(f(x)) = \sqrt{x^2 - x - 2} = \sqrt{(x-2)(x+1)}}$

ii. Write its domain: $\boxed{(-\infty, -1] \cup [2, \infty)}$

iii. Write its range: $\boxed{[0, \infty)}$

5. Solve for the unknowns:

(a) $2 \cdot 4^{x+1} - 10 = 6$. $\quad x =$ **1/2**

Solution:

$$\Rightarrow 2 \cdot 4^{x+1} = 16$$
$$\Rightarrow 4^{x+1} = 8$$
$$\Rightarrow 2(x+1) = 3$$
$$\Rightarrow \boxed{x = 1/2}$$

(b) $\log_3(y+3) - \log_3(y-1) = 2$. $\quad y =$ **3/2**

Solution:

$$\Rightarrow \log_3\left(\frac{y+3}{y-1}\right) = 2$$
$$\Rightarrow \frac{y+3}{y-1} = 9$$
$$\Rightarrow y + 3 = 9y - 9$$
$$\Rightarrow 8y = 12$$
$$\Rightarrow y = \boxed{3/2}$$

Alternative Solution :

$$a = \log_3(y+3), b = \log_3(y-1)$$
$$\Rightarrow a - b = 2,\ 3^a = y+3,\ 3^b = y-1$$
$$\Rightarrow 3^{a-b} = \frac{y+3}{y-1}$$
$$\Rightarrow 3^2 = \frac{y+3}{y-1} \Rightarrow \boxed{y = 3/2}$$

(c) $2^{\log_8(27)} =$ **3**

Solution:

$$2^{\log_8(27)} = (8^{\frac{1}{3}})^{\log_8(27)} = (8^{\log_8(27)})^{\frac{1}{3}} = 27^{\frac{1}{3}} = \boxed{3}$$

Alternative solution:

$$x = \log_8(27) \Rightarrow 8^x = 27 \Rightarrow 2^{3x} = 27 \Rightarrow 2^x = 3 \Rightarrow 2^{\log_8(27)} = \boxed{3}$$

9:15 AM - 10:15 AM version

Solutions

1. From a standard 52-card deck, we are drawing two cards *without replacement*:
(A standard deck has 4 shapes: ♡ hearts, ♢ diamonds, ♣ clubs, ♠ spades, and 13 numbers: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K).

   (a) What is the probability that the first card is a ♢ diamond card? **1/4**

   (b) Given that the first card drawn was a ♢ diamond card, what is the probability the second card is a ♡ heart card?
   13/51

2. City A population doubles every 90 years. City B population doubles every 100 years. In the current year, the two cities have the same population count. When will city A have twice as many inhabitants as city B ? ⇒ number of years: **t = 900**

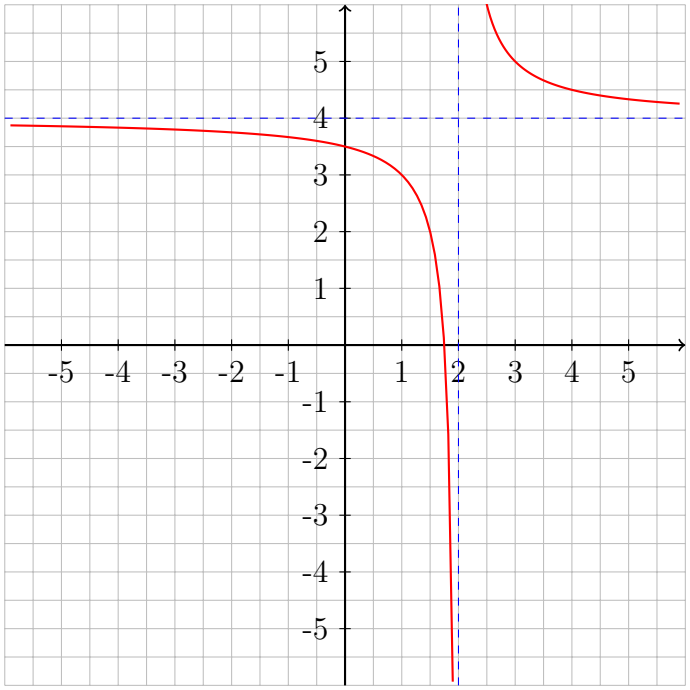
Solution:

$$N \cdot 2^{t/90} = 2 \cdot N \cdot 2^{t/100}$$
$$\Rightarrow \frac{t}{90} = \frac{t}{100} + 1$$
$$\Rightarrow \boxed{t = 900}$$

3. **Sketch** the graph of

$$f(x) = \frac{1}{x-2} + 4$$

on the grid below, and state: the **domain** and **range**, the vertical and horizontal **asymptotes** (in the form $x = a$ or $y = b$) based on the graph.



Solution:

- Domain: $x \in \mathbb{R} \setminus \{2\} \quad \Leftrightarrow x \in (-\infty, 2) \cup (2, \infty)$
- Range: $y \in \mathbb{R} \setminus \{4\} \quad \Leftrightarrow y \in (-\infty, 4) \cup (4, \infty)$
- Vertical asymptote: $x = 2$
- Horizontal asymptote: $y = 4$

4. Let $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 3x$. Compute the following:
   (a) Evaluate $(g \circ f)(14) =$ **4**
   (b) i. Compute $(f \circ g)(x) =$ $\boxed{f(g(x)) = \sqrt{x^2 - 3x + 2} = \sqrt{(x-2)(x-1)}}$
   ii. Domain of $(f \circ g)(x) =$ $\boxed{(-\infty, 1] \cup [2, \infty)}$
   iii. Range of $(f \circ g)(x) =$ $\boxed{[0, \infty)}$

5. Solve the following equations:
   (a) Solve for x in: $3 \cdot 5^{2x} - 8 = 7$ $\Rightarrow x =$ **1/2**

Solution:

$$\Rightarrow 3 \cdot 5^{2x} = 15$$
$$\Rightarrow 5^{2x} = 5$$
$$\Rightarrow 2x = 1$$
$$\Rightarrow x = \boxed{1/2}$$

   (b) Solve for y in: $\log_{10}(y+1) - \log_{10}(y) = 2$ $\Rightarrow y =$ **1/99**

Solution:

$$\Rightarrow \log\left(\frac{y+1}{y}\right) = 2$$
$$\Rightarrow \frac{y+1}{y} = 10^2$$
$$\Rightarrow y + 1 = 100y$$
$$\Rightarrow 99y = 1$$
$$\Rightarrow y = \boxed{1/99}$$

Alternative Solution :

$$a = \log_{10}(y+1), b = \log_{10}(y)$$
$$\Rightarrow a - b = 2,\ 10^a = y + 1,\ 10^b = y$$
$$\Rightarrow 10^{a-b} = \frac{y+1}{y}$$
$$\Rightarrow 10^2 = \frac{y+1}{y} \Rightarrow \boxed{y = 1/99}$$

   (c) $2^{\log_4(7)} =$ $\sqrt{7}$

Solution:

$$2^{\log_4(7)} = 2^{\frac{1}{2}\log_2(7)} = (2^{\log_2(7)})^{\frac{1}{2}} = 7^{\frac{1}{2}} = \boxed{\sqrt{7}}$$

Alternative solution:

$$2^{\log_4(7)} = (4^{\frac{1}{2}})^{\log_4(7)} = (4^{\log_4(7)})^{\frac{1}{2}} = 7^{\frac{1}{2}} = \boxed{\sqrt{7}}$$

Another solution:

$$x = \log_4(7) \Rightarrow 4^x = 7 \Rightarrow 2^{2x} = 7 \Rightarrow 2^x = \sqrt{7} \Rightarrow 2^{\log_4(7)} = \boxed{\sqrt{7}}$$

10:30 AM - 11:30 AM version

Solutions

1. From a standard 52-card deck, we are drawing two cards *without replacement*:
(A standard deck has 4 shapes: ♡ hearts, ♢ diamonds, ♣ clubs, ♠ spades, and 13 numbers: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K).

   (a) What is the probability the first card is a 4? **1/13**

   (b) Given the first card drawn was a 4, what is the probability the second card is a 7? **4/51**

2. Two bacterial cultures have different doubling times. Culture A doubles every 4 hours; Culture B doubles every 6 hours. At time $t = 0$, both cultures have the same number of cells. After how many hours will Culture A have four times as many cells as Culture B?
$\Rightarrow$ number of hours = **t = 24**

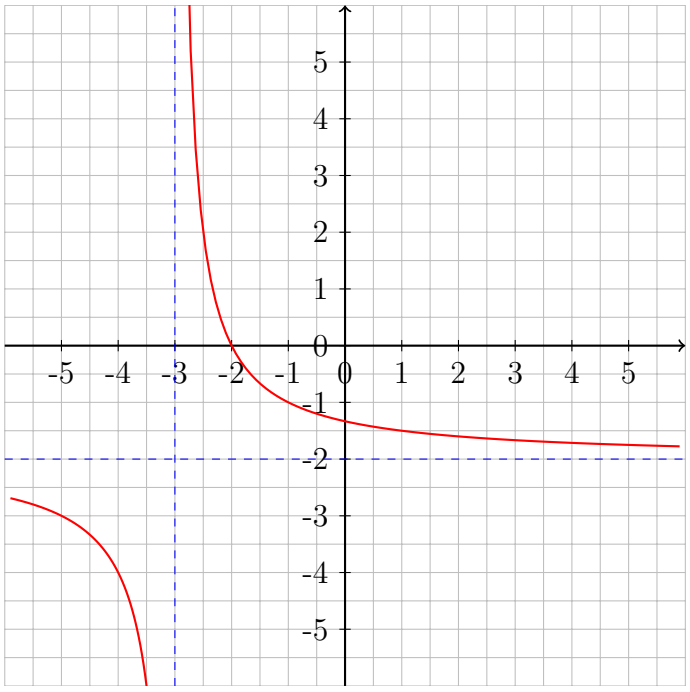
Solution:

$$N \cdot 2^{t/4} = 4 \cdot N \cdot 2^{t/6}$$
$$\Rightarrow \frac{t}{4} = \frac{t}{6} + 2$$
$$\Rightarrow \boxed{t = 24}$$

3. **Sketch** the graph of

$$g(x) = \frac{2}{x+3} - 2$$

on the grid below, and state: the **domain** and **range**, the vertical and horizontal **asymptotes** (in the form $x = a$ or $y = b$) based on the graph.



Solution:

- Domain: $x \in \mathbb{R} \setminus \{-3\} \quad \Leftrightarrow x \in (-\infty, -3) \cup (-3, \infty)$
- Range: $y \in \mathbb{R} \setminus \{-2\} \quad \Leftrightarrow y \in (-\infty, -2) \cup (-2, \infty)$
- Vertical asymptote: $x = -3$
- Horizontal asymptote: $y = -2$

4. Let

$$f(x) = x^2 + 2x \quad \text{and} \quad h(x) = \sqrt{x-3}\,.$$

(a) Evaluate $(f \circ h)(7) =$ **8**

(b) i. Compute $(h \circ f)(x) =$ $\boxed{h(f(x)) = \sqrt{x^2+2x-3} = \sqrt{(x+3)(x-1)}}$

ii. Write its domain: $\boxed{(-\infty, -3] \cup [1, \infty)}$

iii. Write its range: $\boxed{[0, \infty)}$

5. Solve for the unknowns:

(a) $4 \cdot 3^{x-1} - 29 = 7$. $\quad x =$ **3**

Solution:

$$\Rightarrow 4 \cdot 3^{x-1} = 36$$
$$\Rightarrow 3^{x-1} = 9$$
$$\Rightarrow \boxed{x = 3}$$

(b) $\log_2(2y) - \log_2(y-1) = 2$. $\quad y =$ **2**

Solution:

$$\Rightarrow \log_2\left(\frac{2y}{y-1}\right) = 2$$
$$\Rightarrow \frac{2y}{y-1} = 4$$
$$\Rightarrow 2y = 4y - 4$$
$$\Rightarrow \boxed{y = 2}$$

Alternative Solution :

$$a = \log_2(2y), b = \log_2(y-1)$$
$$\Rightarrow a - b = 2,\ 2^a = 2y,\ 2^b = y - 1$$
$$\Rightarrow 2^{a-b} = \frac{2y}{y-1}$$
$$\Rightarrow 2^2 = \frac{2y}{y-1} \Rightarrow \boxed{y = 2}$$

(c) $3^{\log_9(8)} =$ $\sqrt{8}$

Solution:

$$= (9^{\frac{1}{2}})^{\log_9(8)} = (9^{\log_9(8)})^{\frac{1}{2}} = 8^{\frac{1}{2}} = \sqrt{8} = \boxed{2\sqrt{2}}$$

Alternative solution:

$$= 3^{\frac{1}{2}\log_3(8)} = (3^{\log_3(8)})^{\frac{1}{2}} = 8^{\frac{1}{2}} = \sqrt{8} = \boxed{2\sqrt{2}}$$

Another solution:

$$x = \log_9(8) \Rightarrow 9^x = 8 \Rightarrow 3^{2x} = 8 \Rightarrow 3^x = \sqrt{8} \Rightarrow 3^{\log_9(8)} = \boxed{2\sqrt{2}}$$